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RESEARCH REPORT

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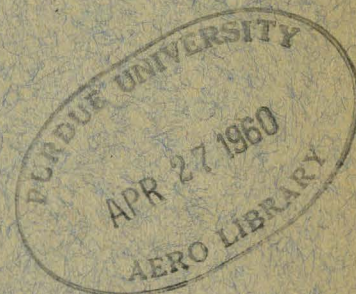
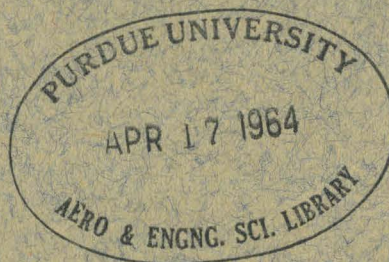
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FUNDAMENTAL STUDIES RELATING TO SYSTEMS ANALYSIS OF SOLID PROPELLANTS

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15 April 1959



② Guggenheim Aeronautical Laboratory
① California Institute of Technology
Pasadena, California

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Fundamental Studies Relating to
Systems Analysis of Solid Propellants

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15 April 1959

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I. INTRODUCTION TO THE STRAIN ANALYSIS OF SOLID PROPELLANT GRAINS

In continuing the investigation of analysis procedures to be used in studying the structural integrity of solid propellant grains, we amplify the content of the first progress report without at this time opening up any new areas. However it is perhaps appropriate to enumerate some of the specific subjects to be presented later. Following the earlier pattern of (1) model representation and (2) tabulated elastic solutions, both of which are supplemented in this report, it is expected to include (3) heat transfer and temperature distributions, (4) engineering analysis, i. e. the practical combination of items (1), (2), and possibly (3) above, (5) failure criteria and strength analysis.

Pending further amplification of the various sections, the interested reader is referred to a general presentation* prepared for the 15th Meeting of the JANAF Solid Propellant Group, June 1959, which outlines some of the background as well as giving an overall, but general, picture of the problem area. It is hoped that the work under this project, as well as that by other investigators will tend to fill in some of the presently existing blank spots.

Turning now to the current work, the earlier discussion of one and two element viscoelastic models is extended to the three and five element representation, and generalized to the infinite number of elements for which case a spectrum distribution is introduced. The emphasis is upon the premise that if the spectrum is known, certain important special analyses and experiments can be conducted.

In the second section, the earlier tabulation of elastic solutions for pressure loadings have been supplemented by similar formulas for interesting cases of thermal and environmental loadings, preparatory for the work engineering analysis to be carried out in the immediate future.

* Williams, M. L.: The Importance of Structural Integrity in Grain Design.

Viscoelastic Model Representation

In order to determine propellant failure criteria from tests on various geometries, it is necessary to know the strained condition of the material prior to rupture. Due to the viscoelastic nature of propellant, both strain magnitudes and rates are important parameters in the prediction of failure. Thus, as a means of attaching significance to the results of failure tests, a suitable theory of strain analysis must first be developed. This phase of the problem was considered in the first progress report and will be extended here.

It was shown in the previous report that an analogy exists between elastic and linear viscoelastic stress (or strain) problems. This mathematical correspondence between the two problems is very useful since viscoelastic stresses and displacements can be obtained from an "associated" elastic solution. The basic difference between the two problems is in the stress-strain behavior. For example, with an elastic material, stress and strain in simple tests (tension, shear, bulk) can be related by a proportionality constant; on the other hand, a linear viscoelastic material was defined to be one in which stress and strain are related by a linear differential equation. As an aid in constructing these equations and determining their physical significance, the concept of mechanical models was introduced. The Voigt and Maxwell models were considered and their behavior was compared qualitatively to that of a typical propellant. The purpose of the following section will be to extend this analysis by considering models which exhibit more general linear viscoelastic behavior.

Three Element Model - Maxwell Element and Spring

The Maxwell model was shown to behave similar to that of a noncross-linked polymer in that unlimited flow (or strain) occurred under a continuously applied stress. In order to describe the behavior of a cross-linked polymer, a spring can be attached in parallel as shown in Figure 7a. Previously, the series arrangement of a spring and dashpot was referred to as a Maxwell model, however when it comprises only part of a more general model, it will be called a Maxwell element.

The modulus of a spring in parallel with one or more Maxwell elements is denoted m_e , the equilibrium modulus. Its physical significance can be seen in Figure 7c, since it represents the long-time modulus of the model under constant strain. In addition, this spring provides complete recovery as shown in Figure 7b.

Another significant parameter is the glass modulus m_g . It is the effective elastic modulus for very short loading times and also corresponds to the effective modulus at temperatures below the glass transition temperature.

Response to the various types of loading shown in Figure 7b, 7c, and 7d is obtained, as before, by integration of the operator equation. The constants of integration are determined by considering the model at rest for $t \leq 0$. A more efficient way of solving these differential equations is to use the Laplace transform⁽⁶⁾, however this technique will not be discussed here.

Five Element Model - Two Maxwell Elements and Spring

In order to fit experimental data over a wider time range than that covered by a three element model, another Maxwell element can be added as illustrated in Figure 8a. In this case, a second order differential equation relates stress and strain. It is important to realize that this does not reflect inertia effects since the roots of the operator equation are always real, which results from having used a model that includes only springs and dashpots.

Response for creep, relaxation, and constant strain rate is seen to be similar to that of the three element model.

The four element model, which is used for noncross-linked polymers, was not discussed since it can be readily obtained from the five element model by setting $m_e = 0$. In addition, composite propellants are usually crosslinked so that the more common condition is $m_e \neq 0$.

For many problems it is expected that the foregoing models will provide a reasonable approximation to propellant response over a limited time interval prior to rupture.

Wiechert or Generalized Maxwell Model

In some cases, it may be necessary to use a model consisting of several Maxwell elements in parallel. This general representation is commonly referred to as a Wiechert model, and is shown in Figure 9 with n Maxwell elements. The equilibrium modulus m_e may or may not be included; if the model is to represent a cross-linked material, then the equilibrium modulus must be used.

The operator equation is quite simple in form if it is left in summation form. However, when it is expanded so that a common denominator is formed, an n th order differential equation is obtained.

As we let n increase, the response of the model becomes more general since there will be additional arbitrary parameters. In fact by letting n tend to infinity, we can represent very generally, linear viscoelastic response. The limit is taken such that the stress remains finite, as indicated in Figure 9.

Instead of having a finite number of discrete parameters we now have introduced an arbitrary function $H(\tau)$, usually called the relaxation spectrum. If $H(\tau)$ and m_e are known, the stress-strain law is completely defined for all types of loading. The many techniques used to determine this function from experimental data will not be given here but are discussed in reference 5.

In order to calculate the stress response to a constant strain input, as in a stress relaxation test, it is necessary to solve this integral equation shown in Figure 9 with $\epsilon = \epsilon_0$. Since the integral represents an infinite sum of terms, the usual integration procedure by partial fractions would yield in principle an infinite number of constants to be determined from initial conditions. However, by means of the Laplace transform the solution for zero initial conditions can be

obtained immediately*. The resulting stress given in Figure 9 (second sheet), is related to the strain by a function of time defined as the relaxation modulus. This modulus evaluated at $t = 0$ is defined as the glass modulus m_g , in accordance with the previous definition when finite element models were considered. The equilibrium modulus m_e takes on the same significance as before.

For constant strain rate, stress can be determined by means of the Laplace transform. The result is seen to be similar to the relaxation modulus, in fact a very interesting relation is shown to exist. Namely, that the slope of the stress-strain curve in a constant strain rate test is equal to the stress relaxation modulus evaluated at (ϵ/R) .

* This will be shown for the reader already familiar with the Laplace transform. If $\theta(t)$ is a function of time and we denote its transform by $\bar{\theta}(s)$, then the transformed operator equation is

$$\bar{\sigma}(s) = \left[m_e + \int_0^{\infty} \frac{H(\tau) S}{(s + \frac{1}{\tau}) \tau} d\tau \right] \bar{\epsilon}(s) \quad (a)$$

in which S is the transform variable and all conditions are taken as zero for $t \leq 0$. Even though this is the transform of an integral expression rather than a finite sum of terms, the standard procedure can be used. The only additional restriction is that the integral converge uniformly. For constant strain,

$$\bar{\epsilon}(s) = \frac{\epsilon_0}{s} \quad (b)$$

Therefore, the transformed equation becomes

$$\bar{\sigma}(s) = \left[\frac{m_e}{s} + \int_0^{\infty} \frac{H(\tau)}{(s + \frac{1}{\tau}) \tau} d\tau \right] \epsilon_0 \quad (c)$$

The integrand can be inverted using the relation:

$$\frac{H(\tau)}{(s + \frac{1}{\tau}) \tau} d\tau = \overline{\left(\frac{H(\tau) e^{-\frac{t}{\tau}}}{\tau} d\tau \right)} \quad (d)$$

Substituting (d) into (c), we obtain the time dependent stress,

$$\sigma(t) = \left[m_e + \int_0^{\infty} \frac{H(\tau) e^{-\frac{t}{\tau}}}{\tau} d\tau \right] \epsilon_0 \quad (e)$$

This relation is independent of the relaxation spectrum and thus depends only on the assumption of linear springs and dashpots. Indeed, this same correspondence exists for the models with a finite number of elements. Such a relation is very useful since data from these two types of tests can be used to check the assumption of linearity.

Kelvin or Generalized Voigt Model

The Maxwell representation of models is not the only one which can be used to describe viscoelastic behavior. The Voigt model discussed in the first progress report can be generalized in the same fashion as done with the Maxwell model. In fact, it turns out that the same linear viscoelastic behavior can be defined by the generalized Voigt model (or Kelvin model) shown in Figure 10*. Therefore, only one method of representation is actually needed to solve stress problems. However, in the determination of model parameters from tests, the type of test generally dictates which form should be used when n is large.

For example, reference to Figure 8 shows that when strain is given as in a stress relaxation test, the stress is a relatively simple

* The equivalence of the Wiechert and Kelvin models can be shown when they have either a finite or infinite number of elements. However, there are a couple of conditions which must be satisfied in order to do this. One is that the finite and infinite element models must both have the same basic behavior in regards to limited or unlimited strain. In particular, if $m_e \neq 0$ in the Wiechert model, then the equivalent Kelvin model must have a spring adjacent to all the dashpots. Similarly, if the Wiechert model represents noncross-linked material ($m_e = 0$), then the equivalent Kelvin model must have a free dashpot in series with the spring k_g . The other condition is that the number of elements in each model must be the same.

To prove that the finite element models are equivalent, it is necessary and sufficient to show that the operator equation for each can be written in identical form. That is, the same derivatives must appear in both numerators and in both denominators. The coefficients of the derivative terms d^n/dt^n will, of course, consist of different parameters. However, by equating the coefficients of the same derivative terms in each model, relations between the parameters of the models are obtained. With an integral representation, the correspondence between model parameters is obtained in the form of integral equations (7).

function of the model parameters. However, the strain equation for a creep test is much more involved. If the Voigt representation had been used the converse would have been true.

Because of the close correspondence between the Wiechert and Kelvin models, the latter will not be discussed in detail. However, equations pertaining to this model are given in Figure 10. It is seen that spring constants are in terms of compliances k_i , rather than their reciprocals m_i ; similarly, the dashpots are defined by fluidity Φ_i instead of the reciprocal η_i . The reason for this nomenclature is seen from the corresponding equations in that the mathematics is analogous to that pertaining to the Wiechert model.

Two simple equations relating parameters of the models are

$$\frac{1}{k_g} = m_g \quad (1)$$

$$\frac{1}{m_e} = k_e \quad (2)$$

Many other more involved expressions relating the infinite element models can be found in the literature on linear viscoelasticity⁽⁷⁾. They are useful in checking the theory by comparing data from various types of tests. It is important to note that such simple relations as (1) and (2) generally do not hold between the time dependent quantities of the models.

For example, it is not true that the relaxation modulus is the reciprocal of the creep compliance.

Summary of Stress-Strain Operator Equations

For easy reference, the operator equations discussed in the first two progress reports are listed in Table I. These expressions, with exception of the two element Maxwell model, are general in the sense that they can be used for tension, shear, and bulk response. As was mentioned previously, the Maxwell model evinces unlimited flow under stress and therefore cannot be used to represent bulk behavior.

When applying these to experimental data or the solving of a stress problem, it is convenient to use different symbols for shear, bulk, and tension. The commonly used operator symbols are given in Table I, along with the analogous elastic moduli. The fractional form P/Q is used by Alfrey⁽⁵⁾, while the combined form corresponds to the notation proposed by the Committee on Nomenclature of the Society of Rheology⁽⁸⁾.

Comparison of the Models

In order to determine the model needed in a particular engineering problem, a comparison between model response and experimental data must be made. As a simple example, we will consider the response for stress relaxation and constant strain rate of the three and five element models relative to the Wiechert model for National Bureau of Standards polyisobutylene (PIB). The relaxation spectrum in tension for polyisobutylene in the glass-to-rubber transition region is⁽⁹⁾

$$H(\tau) = \frac{E_g - E_g}{\Gamma(\beta)} \left(\frac{\tau_0}{\tau} \right)^\beta e^{-\frac{\tau_0}{\tau}} \quad (3)$$

where $\beta \approx 0.68$, $\tau_0 = \text{const}$, $\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx$

The notation E , instead of m , for modulus is used since the data was obtained in a tensile test.

Substituting this spectrum into the expression for the relaxation modulus of the Wiechert model yields the result found from experiment for small strains:

$$\frac{\sigma}{E_g \epsilon_0} = \frac{E(t)}{E_g} = \frac{E_e}{E_g} + \left(1 - \frac{E_e}{E_g} \right) \left(1 + \frac{t}{\tau_0} \right)^{-\beta} \quad (4)$$

Corresponding relaxation moduli for the three and five element models, as shown in Figures 7c and 8c, are:

Three element:
$$\frac{\sigma}{E_g \epsilon_o} = \frac{E(t)}{E_g} = \frac{E_e}{E_g} + \left(1 - \frac{E_e}{E_g}\right) e^{-\frac{t}{\tau_m}} \quad (5)$$

Five element:
$$\frac{\sigma}{E_g \epsilon_o} = \frac{E(t)}{E_g} = \frac{E_e}{E_g} + \frac{E_1}{E_g} e^{-\frac{t}{\tau_1}} + \left(1 - \frac{E_1}{E_g} - \frac{E_e}{E_g}\right) e^{-\frac{t}{\tau_2}} \quad (6)$$

These equations are plotted in Figure 11. E_g was taken to be the same for all models, and also the reasonable assumption that the equilibrium modulus E_e is much less than the glass modulus E_g (i. e. $E_e/E_g \ll 1$) was made. This assumption allows us to neglect E_e/E_g in the above equations as long as we consider only the short time response. The arbitrary parameters of the three and five element models were chosen in order to fit the polyisobutylene modulus for $0 \leq t/\tau_o \leq 10$. The particular values used are:

$$\frac{1}{\tau_m} = \frac{0.30}{\tau_o}, \quad \frac{1}{\tau_1} = \frac{2.11}{\tau_o}, \quad \frac{1}{\tau_2} = \frac{0.20}{\tau_o}, \quad \frac{E_1}{E_g} = 0.25 \quad (7)$$

Stresses in a constant strain rate test are also shown in Figure 11. The governing equations, obtained from Figures 9, 7d, and 8d, are:

Wiechert
$$\frac{\sigma}{RE_g \tau_o} = \frac{1}{\beta-1} + \frac{(1 + \frac{t}{\tau_o})^{1-\beta}}{1-\beta} \quad (8)$$

Three element:
$$\frac{\sigma}{RE_g \tau_m} = \frac{t}{\tau_m} \times \frac{E_e}{E_g} + \left(1 - \frac{E_e}{E_g}\right) \left(1 - e^{-\frac{t}{\tau_m}}\right) \quad (9)$$

Five element:
$$\frac{\sigma}{RE_g \tau_1} = \frac{t}{\tau_1} \times \frac{E_e}{E_g} + \frac{E_1}{E_g} \left(1 - e^{-\frac{t}{\tau_1}}\right) + \frac{\tau_1}{\tau_2} \left(1 - \frac{E_1}{E_g} - \frac{E_e}{E_g}\right) \left(1 - e^{-\frac{t}{\tau_2}}\right) \quad (10)$$

Since these stresses are integrals of the relaxation moduli, it is reasonable that the approximations (9) and (10) should be better than (5) and (6), as indicated by Figure 11.

This comparison illustrates the important point that models containing only a few elements are not sufficient to describe the behavior of polyisobutylene over a broad time scale. Such a condition generally will also exist with a propellant.

However, when using models to solve engineering stress problems, there is a practical mathematical restriction on the number of elements which can be used. In fact, it is usually necessary to use models with no more than three to five elements in order that the algebra does not get out of hand. Thus, it will be most expedient to determine approximate operator equations appropriate for the time scale of interest. Subsequent reports will deal, in part, with this phase of the problem, since it is important that the extent of error is known when an approximate operator equation is used for calculating stresses in grains.

For continuity, the references in the first progress report have been repeated. In addition, the figures are numbered consecutively beginning with the first report.

REFERENCES

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6. For example, see R. V. Churchill, "Modern Operational Mathematics in Engineering". McGraw-Hill Book Company, Inc., New York, 1944.
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9. T. L. Smith, California Institute of Technology, Jet Propulsion Laboratory (personal communication).

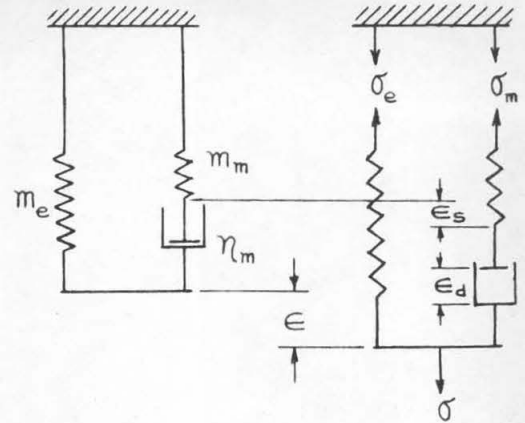
$$\epsilon = \epsilon_s + \epsilon_d$$

$$\sigma = \sigma_e + \sigma_m$$

$$\text{WHERE } \frac{d\epsilon_s}{dt} = \frac{1}{m_m} \frac{d\sigma_m}{dt}$$

$$\frac{d\epsilon_d}{dt} = \frac{\sigma_m}{\eta_m}$$

$$\epsilon = \frac{\sigma_e}{m_e}$$



OPERATOR EQUATION:

$$\sigma(t) = \left[m_e + \frac{m_m \frac{d}{dt}}{\left(\frac{d}{dt} + \frac{1}{\tau_m} \right)} \right] \epsilon(t) \quad \text{OR} \quad \sigma(t) = \frac{m_g \left(\frac{d}{dt} + \frac{m_e}{m_g} \frac{1}{\tau_m} \right)}{\left(\frac{d}{dt} + \frac{1}{\tau_m} \right)} \epsilon(t)$$

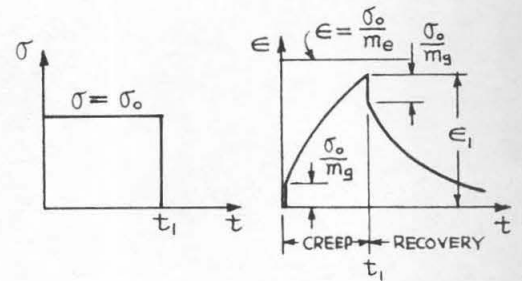
$$\text{WHERE } \tau_m = \frac{\eta_m}{m_m} ; \quad m_m + m_e = m_g$$

(a) MODEL

$$\epsilon = \left[1 - \left(1 - \frac{m_e}{m_g} \right) e^{-\frac{m_e}{m_g} \frac{t}{\tau_m}} \right] \frac{\sigma_o}{m_e} : \text{FOR } 0 < t \leq t_1$$

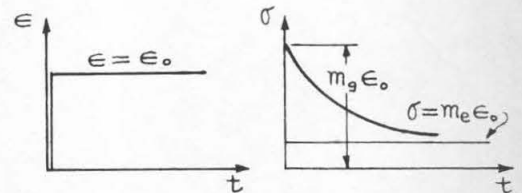
$$\epsilon = \left[\epsilon_1 - \frac{\sigma_o}{m_g} \right] e^{-\frac{m_e}{m_g} \frac{(t-t_1)}{\tau_m}} : \text{FOR } t > t_1$$

(b) CREEP & RECOVERY



$$\sigma = \left[1 + \left(\frac{m_g}{m_e} - 1 \right) e^{-\frac{t}{\tau_m}} \right] m_e \epsilon_o : \text{FOR } t > 0$$

(c) RELAXATION



$$\epsilon = R t \quad (R = \text{STRAIN RATE})$$

$$\sigma = \left[\frac{t}{\tau_m} + \left(\frac{m_g}{m_e} - 1 \right) \left(1 - e^{-\frac{t}{\tau_m}} \right) \right] R m_e \tau_m$$

(d) CONSTANT STRAIN RATE

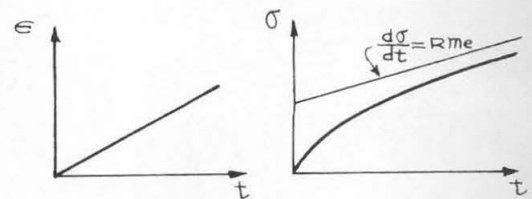


FIG. 7 THREE ELEMENT MODEL: MAXWELL + SPRING

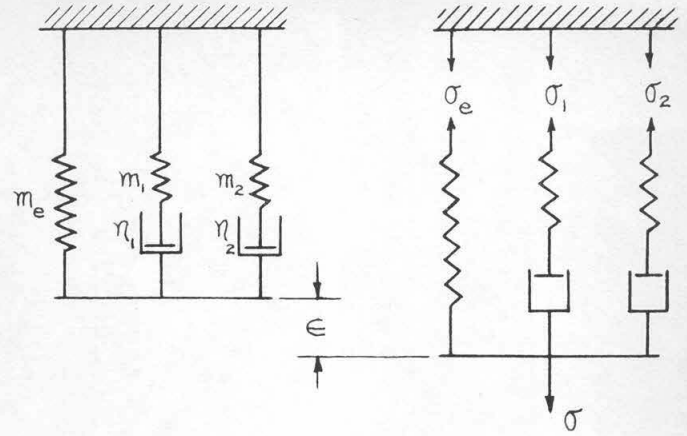
$$\epsilon = \epsilon_{s_1} + \epsilon_{d_1} = \epsilon_{s_2} + \epsilon_{d_2}$$

$$\sigma = \sigma_e + \sigma_1 + \sigma_2$$

WHERE

$$\begin{aligned} \frac{d\epsilon}{dt} &= \left[\frac{1}{m_1} \frac{d}{dt} + \frac{1}{\eta_1} \right] \sigma_1 \\ &= \left[\frac{1}{m_2} \frac{d}{dt} + \frac{1}{\eta_2} \right] \sigma_2 \end{aligned}$$

$$\epsilon = \frac{\sigma_e}{m_e}$$



OPERATOR EQUATION:

$$\sigma(t) = \left[m_e + \frac{m_1 \frac{d}{dt}}{\left(\frac{d}{dt} + \frac{1}{\tau_1} \right)} + \frac{m_2 \frac{d}{dt}}{\left(\frac{d}{dt} + \frac{1}{\tau_2} \right)} \right] \epsilon(t) \text{ OR } \sigma(t) = \frac{m_g \left[\frac{d^2}{dt^2} + \left(\frac{m_e + m_1}{m_g} + \frac{m_e + m_2}{m_g} \right) \frac{d}{dt} + \frac{m_e}{\tau_1 \tau_2} \right] \epsilon(t)}{\left[\frac{d^2}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \frac{d}{dt} + \frac{1}{\tau_1 \tau_2} \right]}$$

WHERE $\tau_1 = \frac{\eta_1}{m_1}$; $\tau_2 = \frac{\eta_2}{m_2}$; $m_g = m_e + m_1 + m_2$

(a) MODEL

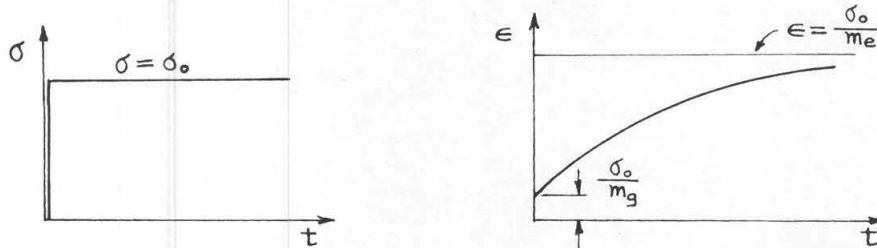
$$\epsilon = \left[\frac{a_0}{\alpha \beta} - \frac{\alpha^2 - a_1 \alpha + a_0}{\alpha(\beta - \alpha)} e^{-\alpha t} - \frac{\beta^2 - a_1 \beta + a_0}{\beta(\alpha - \beta)} e^{-\beta t} \right] \frac{\sigma_0}{m_g} : \text{ FOR } t > 0$$

WHERE

$$a_0 = \frac{1}{\tau_1 \tau_2} ; \quad a_1 = \frac{1}{\tau_1} + \frac{1}{\tau_2}$$

AND α & β ARE FOUND FROM THE FOLLOWING EQUATIONS:

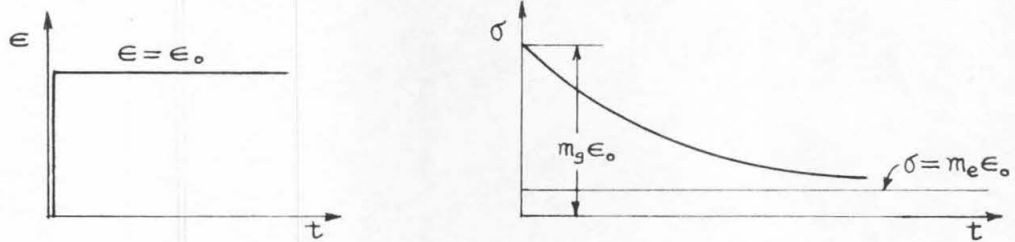
$$\alpha + \beta = \left[\frac{\frac{m_e}{m_g} + \frac{m_1}{m_g}}{\tau_2} + \frac{\frac{m_e}{m_g} + \frac{m_2}{m_g}}{\tau_1} \right] ; \quad \alpha \beta = \frac{\frac{m_e}{m_g}}{\tau_1 \tau_2}$$



(b) CREEP

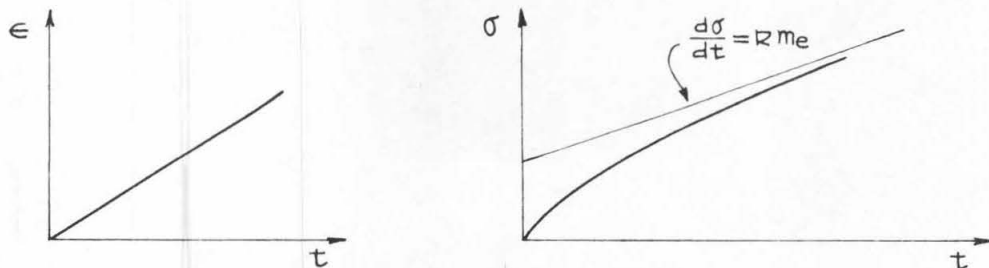
FIG. 8. FIVE ELEMENT MODEL : TWO MAXWELL ELEMENTS + SPRING

$$\sigma = \left[1 + \frac{m_1}{m_e} e^{-\frac{t}{\tau_1}} + \frac{m_2}{m_e} e^{-\frac{t}{\tau_2}} \right] m_e \epsilon_0 : \text{ FOR } t > 0$$

(C) RELAXATION

$$\epsilon = R t$$

$$\sigma = \left[\frac{t}{\tau_1} + \frac{m_1}{m_e} (1 - e^{-\frac{t}{\tau_1}}) + \frac{\tau_2}{\tau_1} \frac{m_2}{m_e} (1 - e^{-\frac{t}{\tau_2}}) \right] R m_e \tau_1$$

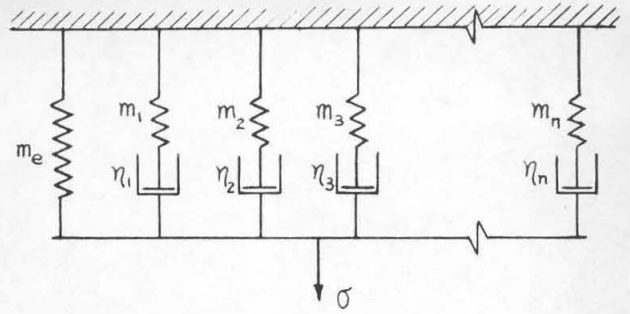
(d) CONSTANT STRAIN RATEFIG. 8. FIVE ELEMENT MODEL : TWO MAXWELL ELEMENTS + SPRING
(Continued)

$$\epsilon = \epsilon_{s_i} + \epsilon_{d_i}$$

$$\sigma = \sigma_e + \sum_{i=1}^n \sigma_i$$

$$\text{WHERE } \frac{d\epsilon}{dt} = \left[\frac{1}{m_i} \frac{d}{dt} + \frac{1}{\eta_i} \right] \sigma_i$$

$$\epsilon = \frac{\sigma_e}{m_e}$$



OPERATOR EQUATION :

$$\sigma(t) = \left[m_e + \sum_{i=1}^n \frac{m_i}{\left(\frac{d}{dt} + \frac{1}{\tau_i} \right)} \right] \epsilon(t)$$

$$\text{WHERE } \tau_i = \frac{\eta_i}{m_i} ; \quad m_g - m_e = \sum_{i=1}^n m_i$$

TO OBTAIN INTEGRAL REPRESENTATION, LET

$$m_i = H(\tau_i) \frac{\Delta \tau_i}{\tau_i}$$

AND TAKE THE LIMIT :

$$\lim_{\substack{n \rightarrow \infty \\ \Delta \tau_i \rightarrow 0 \\ 0 < \tau < \infty}} \sum_{i=1}^n \frac{H(\tau_i) \Delta \tau_i}{\left(\frac{d}{dt} + \frac{1}{\tau_i} \right) \tau_i} = \int_0^{\infty} \frac{H(\tau)}{\left(\frac{d}{dt} + \frac{1}{\tau} \right) \tau} d\tau$$

OPERATOR EQUATION FOR AN INFINITE NUMBER OF ELEMENTS :

$$\sigma(t) = \left[m_e + \int_0^{\infty} \frac{H(\tau)}{\left(\frac{d}{dt} + \frac{1}{\tau} \right) \tau} d\tau \right] \epsilon(t)$$

$H(\tau) \equiv$ RELAXATION SPECTRUM

FIG. 9. WIECHERT MODEL

RELAXATION:

$$\epsilon = \epsilon_0 : \text{ FOR } t > 0$$

$$\sigma(t) = \left[m_e + \int_0^{\infty} \frac{H(\tau) e^{-\frac{t}{\tau}}}{\tau} d\tau \right] \epsilon_0$$

$$\left[m_e + \int_0^{\infty} \frac{H(\tau) e^{-\frac{t}{\tau}}}{\tau} d\tau \right] \equiv m(t) \equiv \text{RELAXATION MODULUS}$$

$$\text{WHERE } m_g - m_e = \int_0^{\infty} \frac{H(\tau)}{\tau} d\tau$$

CONSTANT STRAIN RATE:

$$\epsilon = R t$$

$$\sigma(t) = \left[m_e t + \int_0^{\infty} H(\tau) (1 - e^{-\frac{t}{\tau}}) d\tau \right] R$$

DIFFERENTIATING THIS EQUATION WITH RESPECT TO TIME YIELDS:

$$\frac{d\sigma}{dt} = R m(t) \quad \text{OR} \quad \frac{d\sigma}{d\epsilon} = m\left(\frac{\epsilon}{R}\right)$$

WHERE

σ = STRESS IN CONSTANT STRAIN RATE TEST

FIG. 9. WIECHERT MODEL (Continued)

$$\epsilon = \epsilon_g + \sum_{i=1}^n \epsilon_i$$

$$\sigma = \sigma_{s_i} + \sigma_{d_i}$$

$$\text{WHERE } \frac{d\epsilon_i}{dt} = \phi_i \sigma_{d_i}$$

$$\epsilon_i = k_i \sigma_{s_i}$$

$$\sigma = \left[\frac{1}{\phi_i} \frac{d}{dt} + \frac{1}{k_i} \right] \epsilon_i$$

$$\epsilon_g = k_g \sigma$$

OPERATOR EQUATION:

$$\epsilon(t) = \left[k_g + \sum_{i=1}^n \frac{k_i}{\tau_i \left(\frac{d}{dt} + \frac{1}{\tau_i} \right)} \right] \sigma(t)$$

$$\text{WHERE } \tau_i = \frac{k_i}{\phi_i}; \quad k_e - k_g = \sum_{i=1}^n k_i$$

TO OBTAIN INTEGRAL REPRESENTATION, LET

$$k_i = L(\tau_i) \frac{\Delta_i \tau}{\tau_i}$$

AND TAKE THE LIMIT:

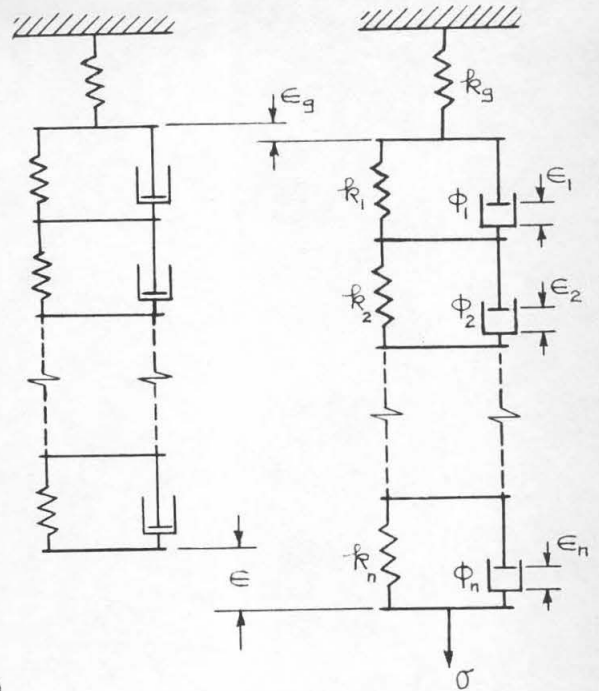
$$\lim_{\substack{n \rightarrow \infty \\ \Delta_i \tau \rightarrow 0 \\ 0 < \tau < \infty}} \sum_{i=1}^n \frac{L(\tau_i) \Delta_i \tau}{\tau_i \left(\frac{d}{dt} + \frac{1}{\tau_i} \right) \tau_i} = \int_0^{\infty} \frac{L(\tau)}{\left(\frac{d}{dt} + \frac{1}{\tau} \right) \tau^2} d\tau$$

OPERATOR EQUATION FOR AN INFINITE NUMBER OF ELEMENTS:

$$\epsilon(t) = \left[k_g + \int_0^{\infty} \frac{L(\tau)}{\left(\frac{d}{dt} + \frac{1}{\tau} \right) \tau^2} d\tau \right] \sigma(t)$$

$L(\tau) \equiv$ RETARDATION SPECTRUM

FIG. 10. KELVIN MODEL



CREEP:

$$\sigma = \sigma_0 : \text{ FOR } t > 0$$

$$\epsilon(t) = \left[k_g + \int_0^\infty \frac{L(\tau)(1 - e^{-\frac{t}{\tau}})}{\tau} d\tau \right] \sigma_0$$

$$\left[k_g + \int_0^\infty \frac{L(\tau)(1 - e^{-\frac{t}{\tau}})}{\tau} d\tau \right] \equiv k(t) \equiv \text{CREEP COMPLIANCE}$$

WHERE

$$k_e - k_g = \int_0^\infty \frac{L(\tau)}{\tau} d\tau$$

CONSTANT STRESS RATE:

$$\sigma = R t$$

$$\epsilon(t) = \left\{ k t + \int_0^\infty L(\tau) \left[\frac{t}{\tau} - (1 - e^{-\frac{t}{\tau}}) \right] d\tau \right\} R$$

DIFFERENTIATING THIS EQUATION WITH RESPECT TO TIME YIELDS:

$$\frac{d\epsilon}{dt} = R k(t) \quad \text{OR} \quad \frac{d\epsilon}{d\sigma} = k\left(\frac{\sigma}{R}\right)$$

WHERE

 ϵ = STRAIN IN CONSTANT STRESS RATE TESTFIG. 10. KELVIN MODEL (Continued).

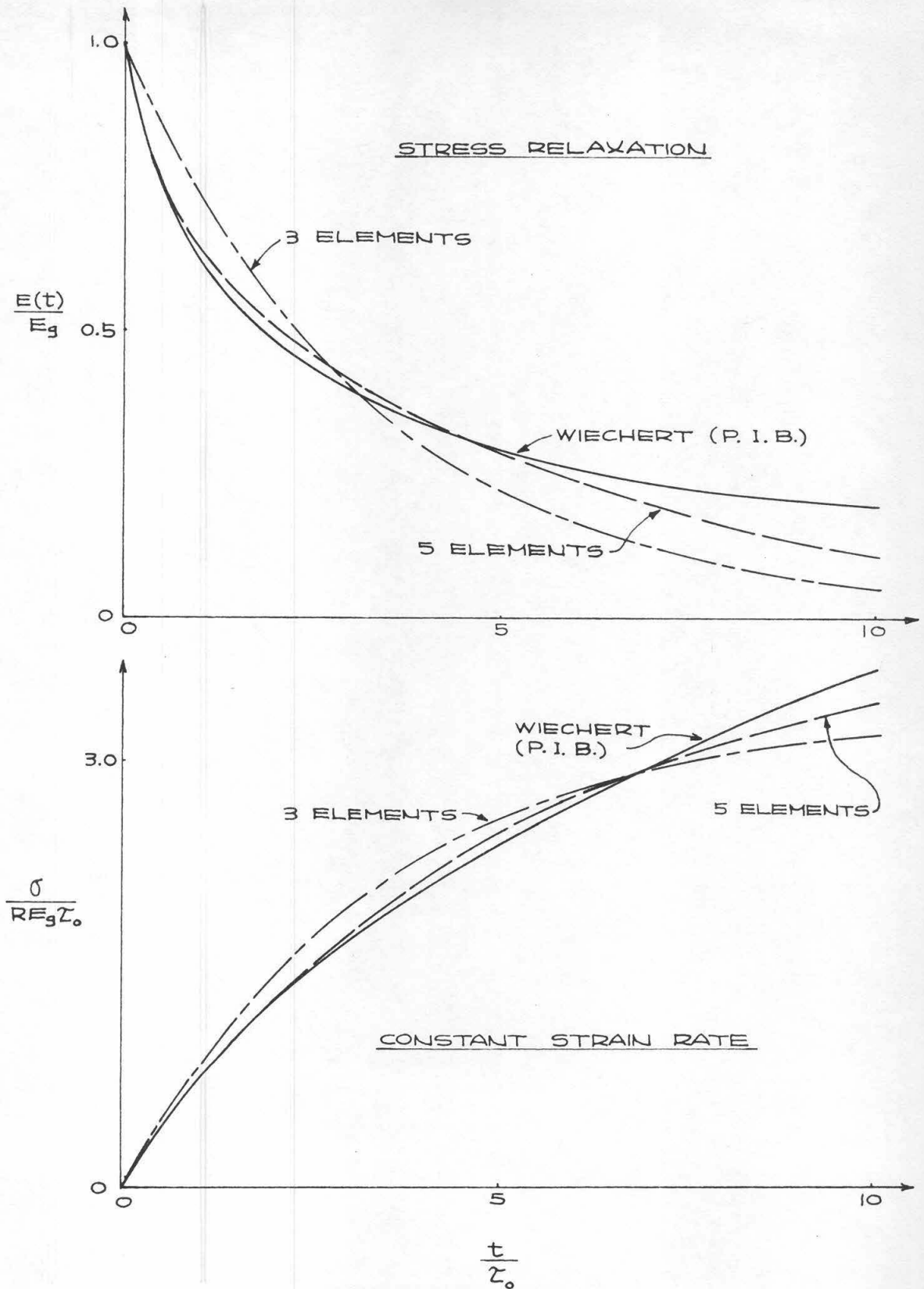


FIG. 11. COMPARISON OF MODELS FOR STRESS RELAXATION AND CONSTANT STRAIN RATE TESTS

TABLE I

SUMMARY OF STRESS-STRAIN OPERATOR EQUATIONS

GENERAL FORM FOR TENSION, SHEAR, AND BULK RESPONSE:

$$\sigma(t) = m(P) \epsilon(t)$$

or
$$\epsilon(t) = k(P) \sigma(t)$$

defining
$$\frac{d}{dt} = P ;$$

$$m(P) = \text{operational form of modulus}$$

$$k(P) = \text{operational form of compliance}$$

$$m(P) = \frac{1}{k(P)}$$

Two Element Models:

(a) Voigt;
$$m(P) = \tau_v m_v \left(P + \frac{1}{\tau_v} \right)$$

(b) Maxwell;
$$m(P) = \frac{m_m P}{\left(P + \frac{1}{\tau_m} \right)}$$

Three Element Model: (Maxwell Representation)

$$m(P) = \frac{m_g \left(P + \frac{m_e}{m_g} \frac{1}{\tau_m} \right)}{\left(P + \frac{1}{\tau_m} \right)}$$

TABLE I. (cont'd.)

Five Element Model: (Maxwell Representation)

$$m(P) = \frac{m_g \left[P^2 + \left(\frac{m_e}{m_g} + \frac{m_1}{m_g} \right) \frac{1}{\tau_1} + \frac{m_e}{m_g} + \frac{m_2}{m_g} \right] P + \frac{m_e}{\tau_1 \tau_2}}{\left[P^2 + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} \right) P + \frac{1}{\tau_1 \tau_2} \right]}$$

Wiechert Model: (Generalized Maxwell)

(a) Discrete Spectrum

$$m(P) = m_e + \sum_{i=1}^n \frac{m_i P}{\left(P + \frac{1}{\tau_i} \right)}$$

(b) Continuous Spectrum

$$m(P) = m_e + \int_0^{\infty} \frac{H(\tau) P}{\left(P + \frac{1}{\tau} \right) \tau} d\tau$$

Kelvin Model: (Generalized Voigt)

(a) Discrete Spectrum

$$k(P) = k_g + \sum_{i=1}^n \frac{K_i}{\left(P + \frac{1}{\tau_i} \right) \tau_i}$$

(b) Continuous Spectrum

$$k(P) = k_g + \int_0^{\infty} \frac{L(\tau)}{\left(P + \frac{1}{\tau} \right) \tau^2} d\tau$$

Symbols for Particular Application of Operator Equations

Type of Deformation	Elastic Constants	Viscoelastic Operators (Alfrey)		Viscoelastic Operators (Society of Rheology)	
		modulus	compliance	modulus	compliance
Simple Tension	E	$\frac{Q''}{P''}$	$\frac{P''}{Q''}$	E(P)	D(P)
Bulk	K	$\frac{Q'}{P'}$	$\frac{P'}{Q'}$	K(P)	B(P)
Shear	G	$\frac{Q}{P}$	$\frac{P}{Q}$	G(P)	J(P)

II. ELASTIC SOLUTIONS FOR CYLINDERS

This section is a continuation of the handbook of elastic solutions given in the first progress report. It includes an additional pressure solution which incorporates the axial tension due to internal pressure, various solutions for arbitrary thermal distributions, and an elementary solution of axial shearing due to gravity loading. These solutions are outlined in several references and are based on small deformation elasticity theory which may be considered of practical engineering use up to ten or twenty percent strain.

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Temperature-Solid or Hollow Cylinder-With Case- Plane Stress	$\Delta T-4, 5$
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DEFINITIONS OF SYMBOLS AND TERMS

a	- inside radius of cylindrical propellant grain
b	- outside radius of cylindrical propellant grain
c	- outside radius of case, as subscript for case properties
E	- Young's modulus
e	- relative volume change: $\frac{\Delta V}{V} = \epsilon_r + \epsilon_\theta + \epsilon_z$
g	- acceleration due to gravity 32.17 fps
h	- case thickness $\ll b$
log	- natural logarithm, base e = 2.71
plane strain	- no strain in axial direction, $\epsilon_z = 0$
plane stress	- no stress in axial direction, $\sigma_z = 0$
p	- pressure, psi.
r	- radial coordinate
steady flow	- temperature distribution constant with time
T	- temperature change from reference or initial temperature
u	- radial displacement
w	- axial displacement
z	- axial coordinate
α	- coefficient of thermal expansion
γ	- shear strain
ϵ_c	- case to propellant modulus ratio = $2bhE_c / [(b^2 - a^2)E]$
ϵ	- normal strain
θ	- tangential coordinate of the cylindrical system
μ	- shear modulus of elasticity = $E / [2(1 + \nu)]$
λ	- Lamé constant = $\nu E / [(1 + \nu)(1 - 2\nu)]$
ν	- Poisson's ratio
ρ	- specific density - lbs/in ³
σ	- normal stress
τ	- shear stress
ϕ	- stress function

PRESSURE - HOLLOW CYLINDER - INTERNAL PRESSURE - WITH CASE AND ENDS BONDED

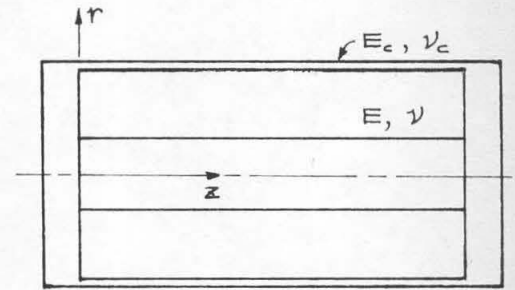
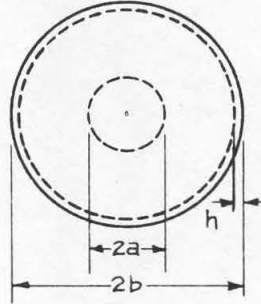
BOUNDARY CONDITIONS:

$$\sigma_r = -p_i : r = a$$

$$\sigma_r = 0 : r = c$$

$$u = u_c : r = b$$

$$w = w_c : z = 0, l$$



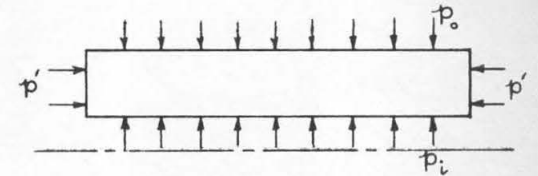
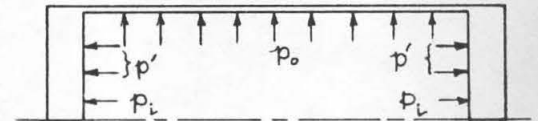
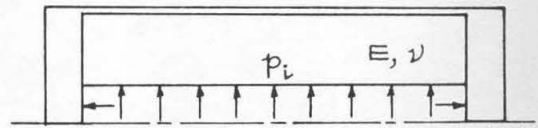
STRESSES IN HOLLOW CYLINDER (PROPELLANT):

$$\sigma_{(r)} = \pm \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2},$$

p_o = PRESSURE BETWEEN CASE & PROPELLANT

$$\sigma_z = -p',$$

p' = PRESSURE BETWEEN ENDS & PROPELLANT



$$p_o = \frac{2(1-\nu^2)\epsilon_c^2 + \{2-\nu+\nu_c(1-2\nu)\}\epsilon_c}{(1+\nu)\left\{(1-2\nu)\frac{b^2}{a^2}+1\right\}\epsilon_c^2 + \left\{[3-\nu(1+4\nu_c)]\frac{b^2}{a^2}+(1+\nu)\right\}\epsilon_c + 2(1-\nu_c^2)\frac{b^2}{a^2}} p_i$$

$$p' = \frac{2\nu\left\{(1+\nu)\left(\frac{b^2}{a^2}-1\right)\right\}\epsilon_c^2 + \left\{[-(1+3\nu)+2\nu_c(2+\nu)]\frac{b^2}{a^2}-(1+\nu)\right\}\epsilon_c - 2(1-\nu_c^2)\frac{b^2}{a^2}}{\left[\frac{b^2}{a^2}-1\right]\left\{(1+\nu)\left\{(1-2\nu)\frac{b^2}{a^2}+1\right\}\epsilon_c^2 + \left\{[3-\nu(1+4\nu_c)]\frac{b^2}{a^2}+(1+\nu)\right\}\epsilon_c + 2(1-\nu_c^2)\frac{b^2}{a^2}\right}} p_i$$

WHERE

$$\epsilon_c = \frac{2bh E_c}{(b^2 - a^2) E} \quad (\text{THIN CASE})$$

STRAINS & DISPLACEMENTS IN PROPELLANT:

$$\epsilon_{(r)} = \frac{1}{E} \left[\pm (1+\nu) \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \frac{1}{r^2} + (1-\nu) \frac{p_i a^2 - p_o b^2}{b^2 - a^2} + \nu p' \right]$$

$$\epsilon_z = -\frac{1}{E} \left[p' + 2\nu \left(\frac{p_i a^2 - p_o b^2}{b^2 - a^2} \right) \right]$$

$$u = \frac{1}{E} \left[-(1+\nu) \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \frac{1}{r} + (1-\nu) \frac{p_i a^2 - p_o b^2}{b^2 - a^2} r + \nu p' r \right]$$

$$w = -\frac{1}{E} \left[p' + 2\nu \left(\frac{p_i a^2 - p_o b^2}{b^2 - a^2} \right) \right] z$$

STRESSES IN CASE:

$$\sigma_{(r)} = \frac{b p_o}{2h} \left(1 \mp \frac{b^2}{r^2} \right) \quad (\text{THIN CASE})$$

$$\sigma_z = p'' = \frac{(b^2 - a^2) p' + a^2 p_i}{2bh}$$

STRAINS & DISPLACEMENTS IN CASE:

$$\epsilon_{(r)} = \frac{1}{E_c} \left[\mp (1+\nu_c) \frac{b^3 p_o}{2h} \frac{1}{r^2} + (1-\nu_c) \frac{b p_o}{2h} - \nu_c p'' \right]$$

$$\epsilon_z = \frac{1}{E_c} \left[p'' - \nu_c \frac{b p_o}{h} \right]$$

$$u_c = \frac{1}{E_c} \left[(1+\nu_c) \frac{b^3 p_o}{2h} \frac{1}{r} + (1-\nu_c) \frac{b p_o}{2h} r - \nu_c p'' r \right]$$

$$w_c = \frac{1}{E_c} \left[p'' - \nu_c \frac{b p_o}{h} \right] z$$

ASSUMPTIONS:

1. ENDS & CASE BONDED TO PROPELLANT.
2. UNIFORM END EFFECTS WITH END PLATES TRANSMITTING FORCES FROM p_i & p' INTO THIN CYLINDRICAL CASE IN TENSION.
3. NO BENDING EFFECTS INCLUDED FROM ENDS; ASSUMING ONLY HOOP (σ_θ) & AXIAL (σ_z) STRESSES UNIFORM IN CASE UP TO ENDS.

STRESS ON INNER RADIUS OF PROPELLANT:

$$\sigma_{\theta}|_{r=a} = \frac{1}{\frac{b^2}{a^2}-1} \left[\frac{(1+\nu)\left\{(1-2\nu)\frac{b^4}{a^4}-2(1-\nu)\frac{b^2}{a^2}+1\right\}\epsilon_c^2 + \left\{[3-\nu(1+4\nu_c)]\frac{b^4}{a^4}+2(\nu-\nu_c)\frac{b^2}{a^2}+(1+\nu)\right\}\epsilon_c + 2(1-\nu_c^2)(1+\frac{b^2}{a^2})\frac{b^2}{a^2}}{(1+\nu)\left\{(1-2\nu)\frac{b^2}{a^2}+1\right\}\epsilon_c^2 + \left\{[3-\nu(1+4\nu_c)]\frac{b^2}{a^2}+(1+\nu)\right\}\epsilon_c + 2(1-\nu_c^2)\frac{b^2}{a^2}} \right] p_i$$

STRAIN & DISPLACEMENT ON INNER RADIUS:

$$\begin{aligned} \epsilon_{\theta}|_{r=a} &= \frac{u}{a}|_{r=a} \\ &= \frac{1}{E(\frac{b^2}{a^2}-1)} \left[\frac{(1+\nu)^2(1-2\nu)(\frac{b^2}{a^2}-1)^2\epsilon_c^2 + (1+\nu)\left\{[3-\nu(1+4\nu_c)]\frac{b^4}{a^4}-[\nu+2\nu_c(1-3\nu)]\frac{b^2}{a^2}+(1-2\nu)\right\}\epsilon_c + 2(1-\nu_c^2)\left[(1+\nu)\frac{b^2}{a^2} + (1-2\nu)\right]\frac{b^2}{a^2}}{(1+\nu)\left\{(1-2\nu)\frac{b^2}{a^2}+1\right\}\epsilon_c^2 + \left\{[3-\nu(1+4\nu_c)]\frac{b^2}{a^2}+(1+\nu)\right\}\epsilon_c + 2(1-\nu_c^2)\frac{b^2}{a^2}} \right] p_i \end{aligned}$$

STRAIN & DISPLACEMENT ON OUTER RADIUS:

$$\begin{aligned} \epsilon_{\theta}|_{r=b} &= \frac{u}{b}|_{r=b} \\ &= \frac{1}{E(\frac{b^2}{a^2}-1)} \left[\frac{\left\{(1+\nu)[4(1-\nu)-\nu_c]\frac{b^2}{a^2}-\nu_c(1-2\nu)\right\}\epsilon_c + 2(2-\nu)(1-\nu_c^2)\frac{b^2}{a^2}}{(1+\nu)\left\{(1-2\nu)\frac{b^2}{a^2}+1\right\}\epsilon_c^2 + \left\{[3-\nu(1+4\nu_c)]\frac{b^2}{a^2}+(1+\nu)\right\}\epsilon_c + 2(1-\nu_c^2)\frac{b^2}{a^2}} \right] p_i \end{aligned}$$

AXIAL STRAIN & DISPLACEMENT:

$$\begin{aligned} \epsilon_x|_{z=l} &= \frac{w}{l}|_{z=l} \\ &= \frac{1}{E(\frac{b^2}{a^2}-1)} \left[\frac{(1+\nu)\left\{[1-4\nu_c(1-\nu)]\frac{b^2}{a^2}+(1-2\nu)\right\}\epsilon_c + 2(1-2\nu)(1-\nu_c^2)\frac{b^2}{a^2}}{(1+\nu)\left\{(1-2\nu)\frac{b^2}{a^2}+1\right\}\epsilon_c^2 + \left\{[3-\nu(1+4\nu_c)]\frac{b^2}{a^2}+(1+\nu)\right\}\epsilon_c + 2(1-\nu_c^2)\frac{b^2}{a^2}} \right] p_i \end{aligned}$$

$$\Delta T - 1$$

THERMOELASTIC EQUATIONS

EQUILIBRIUM:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

STRESS FUNCTION:

$$\sigma_r = \frac{\phi}{r}, \quad \sigma_\theta = \frac{\partial \phi}{\partial r}$$

STRESS - STRAIN:

$$\epsilon_r = \frac{1}{E} [\sigma_r - \nu(\sigma_\theta + \sigma_z)] + \alpha T$$

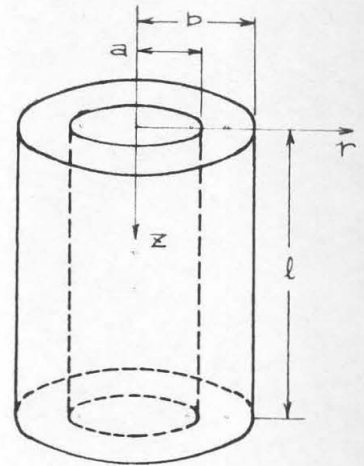
$$\epsilon_\theta = \frac{1}{E} [\sigma_\theta - \nu(\sigma_r + \sigma_z)] + \alpha T$$

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_r + \sigma_\theta)] + \alpha T$$

$$\left. \begin{aligned} \sigma_r &= \lambda e + 2\mu \epsilon_r - \frac{\alpha E T}{1-2\nu} \\ \sigma_\theta &= \lambda e + 2\mu \epsilon_\theta - \frac{\alpha E T}{1-2\nu} \\ \sigma_z &= \lambda e + 2\mu \epsilon_z - \frac{\alpha E T}{1-2\nu} \end{aligned} \right\}$$

$$e = \epsilon_r + \epsilon_\theta + \epsilon_z$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$



COMPATIBILITY:

$$r \frac{d\epsilon_\theta}{dr} + \epsilon_\theta - \epsilon_r = 0$$

PLANE STRESS:

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{\phi}{r^2} = -\alpha E \frac{\partial T}{\partial r}$$

$$\phi = -\frac{\alpha E}{r} \int_a^r T r dr + \frac{C_1 r}{2} + \frac{C_2}{r}$$

PLANE STRAIN:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{\phi}{r^2} = -\frac{\alpha E}{1-\nu} \frac{\partial T}{\partial r}$$

$$\phi = -\frac{\alpha E}{1-\nu} \frac{1}{r} \int_a^r T r dr + \frac{C_1 r}{2} + \frac{C_2}{r}$$

C_1 & C_2 ARE DETERMINED FROM BOUNDARY CONDITIONS.

TEMPERATURE - SOLID OR HOLLOW CYLINDER - NO CASEPLANE STRESS:BOUNDARY CONDITIONS:

$$\sigma_r = 0: r = a, b$$

$$\sigma_z = 0$$

STRESSES:

$$\sigma_r = \frac{\alpha E}{(b^2 - a^2)} \left(1 - \frac{a^2}{r^2} \right) \int_a^b T r dr - \frac{\alpha E}{r^2} \int_a^r T r dr$$

$$\sigma_\theta = \frac{\alpha E}{b^2 - a^2} \left(1 + \frac{a^2}{r^2} \right) \int_a^b T r dr + \frac{\alpha E}{r^2} \int_a^r T r dr - \alpha E T$$

STRAINS & DISPLACEMENTS:

$$\epsilon_r = \frac{\alpha}{b^2 - a^2} \left[(1 - \nu) - (1 + \nu) \frac{a^2}{r^2} \right] \int_a^b T r dr - \frac{(1 + \nu) \alpha}{r^2} \int_a^r T r dr + (1 + \nu) \alpha T$$

$$\epsilon_\theta = \frac{\alpha}{b^2 - a^2} \left[(1 - \nu) + (1 + \nu) \frac{a^2}{r^2} \right] \int_a^b T r dr + \frac{(1 + \nu) \alpha}{r^2} \int_a^r T r dr$$

$$\epsilon_z = - \frac{2\nu \alpha}{b^2 - a^2} \int_a^b T r dr + (1 + \nu) \alpha T$$

$$u = \frac{\alpha}{b^2 - a^2} \left[(1 - \nu) r + (1 + \nu) \frac{a^2}{r} \right] \int_a^b T r dr + \frac{(1 + \nu) \alpha}{r} \int_a^r T r dr$$

$$w = \left[- \frac{2\nu \alpha}{b^2 - a^2} \int_a^b T r dr + (1 + \nu) \alpha T \right] z$$

TEMPERATURE CONSTANT:

$$\sigma_r = \sigma_\theta = 0$$

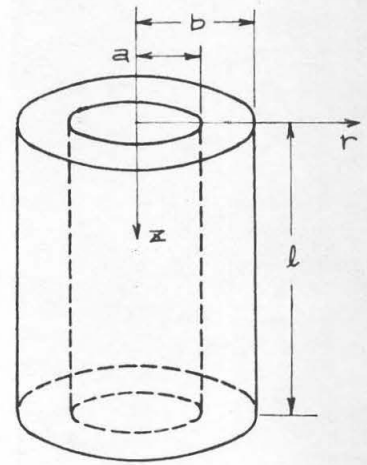
$$\epsilon_r = \epsilon_\theta = \epsilon_z = \alpha T$$

$$u = \alpha T r$$

$$w = \alpha T z$$

ASSUMPTION:

1. NO END EFFECTS.



$$\Delta T - 3$$

TEMPERATURE - SOLID OR HOLLOW CYLINDER - NO CASE

PLANE STRAIN:

BOUNDARY CONDITIONS:

$$\sigma_r = 0 : r = a, b$$

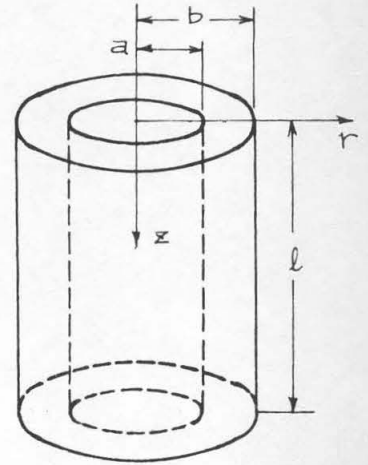
$$w = 0$$

STRESSES:

$$\sigma_r = \frac{\alpha E}{(1-\nu)(b^2-a^2)} \left(1 - \frac{a^2}{r^2}\right) \int_a^b T r dr - \frac{\alpha E}{(1-\nu) r^2} \int_a^r T r dr$$

$$\sigma_\theta = \frac{\alpha E}{(1-\nu)(b^2-a^2)} \left(1 + \frac{a^2}{r^2}\right) \int_a^b T r dr + \frac{\alpha E}{(1-\nu) r^2} \int_a^r T r dr - \frac{\alpha E T}{1-\nu}$$

$$\sigma_z = \frac{2\nu\alpha E}{(1-\nu)(b^2-a^2)} \int_a^b T r dr - \frac{\alpha E T}{1-\nu}$$



STRAINS & DISPLACEMENTS:

$$\epsilon_r = \frac{(1+\nu)\alpha}{(1-\nu)(b^2-a^2)} \left[\left(1-2\nu\right) - \frac{a^2}{r^2} \right] \int_a^b T r dr - \frac{(1+\nu)\alpha}{(1-\nu) r^2} \int_a^r T r dr + \frac{1+\nu}{1-\nu} \alpha T$$

$$\epsilon_\theta = \frac{(1+\nu)\alpha}{(1-\nu)(b^2-a^2)} \left[\left(1-2\nu\right) + \frac{a^2}{r^2} \right] \int_a^b T r dr + \frac{(1+\nu)\alpha}{(1-\nu) r^2} \int_a^r T r dr$$

$$u = \frac{(1+\nu)\alpha}{(1-\nu)(b^2-a^2)} \left[\left(1-2\nu\right) r + \frac{a^2}{r} \right] \int_a^b T r dr + \frac{(1+\nu)\alpha}{(1-\nu) r} \int_a^r T r dr$$

TEMPERATURE CONSTANT:

$$\sigma_r = \sigma_\theta = 0$$

$$\sigma_z = -\alpha E T$$

$$\epsilon_r = \epsilon_\theta = (1+\nu)\alpha T$$

$$u = (1+\nu)\alpha T r$$

ASSUMPTION:

1. NO END EFFECTS.

TEMPERATURE - SOLID OR HOLLOW CYLINDER - WITH CASE

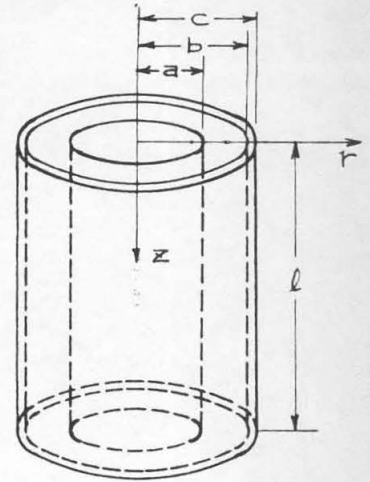
PLANE STRESS:

BOUNDARY CONDITIONS:

$$\sigma_r = 0 : r = a, c$$

$$\sigma_z = 0$$

$$u|_{r=b} = u_c|_{r=b} \quad (\text{ASSUME } w \approx w_c)$$



STRESSES IN PROPELLANT:

$$\sigma_r = -\frac{b^2 p'}{b^2 - a^2} \left(1 - \frac{a^2}{r^2}\right) + \frac{\alpha E}{b^2 - a^2} \left(1 - \frac{a^2}{r^2}\right) \int_a^b T r dr - \frac{\alpha E}{r^2} \int_a^r T r dr$$

$$\sigma_\theta = -\frac{b^2 p'}{b^2 - a^2} \left(1 + \frac{a^2}{r^2}\right) + \frac{\alpha E}{b^2 - a^2} \left(1 + \frac{a^2}{r^2}\right) \int_a^b T r dr + \frac{\alpha E}{r^2} \int_a^r T r dr - \alpha E T$$

STRAINS & DISPLACEMENTS IN PROPELLANTS:

$$\epsilon_r = -\frac{b^2 p'}{E(b^2 - a^2)} \left[(1 - \nu) - (1 + \nu) \frac{a^2}{r^2} \right] + \frac{\alpha}{b^2 - a^2} \left[(1 - \nu) - (1 + \nu) \frac{a^2}{r^2} \right] \int_a^b T r dr - \frac{(1 + \nu) \alpha}{r^2} \int_a^r T r dr + (1 + \nu) \alpha T$$

$$\epsilon_\theta = -\frac{b^2 p'}{E(b^2 - a^2)} \left[(1 - \nu) + (1 + \nu) \frac{a^2}{r^2} \right] + \frac{\alpha}{b^2 - a^2} \left[(1 - \nu) + (1 + \nu) \frac{a^2}{r^2} \right] \int_a^b T r dr + \frac{(1 + \nu) \alpha}{r^2} \int_a^r T r dr$$

$$\epsilon_z = \frac{2 b^2 \nu p'}{E(b^2 - a^2)} - \frac{2 \nu \alpha}{b^2 - a^2} \int_a^b T r dr + (1 + \nu) \alpha T$$

$$u = -\frac{b^2 p'}{E(b^2 - a^2)} \left[(1 - \nu) r + (1 + \nu) \frac{a^2}{r} \right] + \frac{\alpha}{b^2 - a^2} \left[(1 - \nu) r + (1 + \nu) \frac{a^2}{r} \right] \int_a^b T r dr + \frac{(1 + \nu) \alpha}{r} \int_a^r T r dr$$

$$w = \left[\frac{2 b^2 \nu p'}{E(b^2 - a^2)} - \frac{2 \nu \alpha}{b^2 - a^2} \int_a^b T r dr + (1 + \nu) \alpha T \right] z$$

$$p' = \frac{\frac{2 \alpha}{(b^2 - a^2)} \int_a^b T r dr - \alpha_c T}{\frac{[(1 - \nu) b^2 + (1 + \nu) a^2]}{(b^2 - a^2) E} + \frac{b}{h E_c}}$$

TEMPERATURE - SOLID OR HOLLOW CYLINDER - WITH CASEPLANE STRESS CONT'D.:STRESSES IN CASE:

$$\sigma_r = -\frac{b^2 p'}{c^2 - b^2} \left(1 - \frac{c^2}{r^2}\right) + \frac{\alpha_c E_c}{c^2 - b^2} \left(1 - \frac{b^2}{r^2}\right) \int_b^c T r dr - \frac{\alpha_c E_c}{r^2} \int_b^r T r dr$$

$$\sigma_\theta = -\frac{b^2 p'}{c^2 - b^2} \left(1 + \frac{c^2}{r^2}\right) + \frac{\alpha_c E_c}{c^2 - b^2} \left(1 + \frac{b^2}{r^2}\right) \int_b^c T r dr + \frac{\alpha_c E_c}{r^2} \int_b^r T r dr - \alpha_c E_c T$$

STRAINS & DISPLACEMENTS IN CASE:

$$\epsilon_r = \frac{b^2 p'}{E_c (c^2 - b^2)} \left[(1 - \nu_c) - (1 + \nu_c) \frac{c^2}{r^2} \right] + \frac{\alpha_c}{c^2 - b^2} \left[(1 - \nu_c) - (1 + \nu_c) \frac{b^2}{r^2} \right] \int_b^c T r dr +$$

$$- \frac{(1 + \nu_c) \alpha_c}{r^2} \int_b^r T r dr + (1 + \nu_c) \alpha_c T$$

$$\epsilon_\theta = \frac{b^2 p'}{E_c (c^2 - b^2)} \left[(1 - \nu_c) + (1 + \nu_c) \frac{c^2}{r^2} \right] + \frac{\alpha_c}{c^2 - b^2} \left[(1 - \nu_c) + (1 + \nu_c) \frac{b^2}{r^2} \right] \int_b^c T r dr$$

$$+ \frac{(1 + \nu_c) \alpha_c}{r^2} \int_b^r T r dr$$

$$\epsilon_z = -\frac{2 \nu_c b^2 p'}{E_c (c^2 - b^2)} - \frac{2 \nu_c \alpha_c}{c^2 - b^2} \int_b^c T r dr + (1 + \nu_c) \alpha_c T$$

$$u_c = \frac{b^2 p'}{E_c (c^2 - b^2)} \left[(1 - \nu_c) r + (1 + \nu_c) \frac{c^2}{r} \right] + \frac{\alpha_c}{c^2 - b^2} \left[(1 - \nu_c) r + (1 + \nu_c) \frac{b^2}{r} \right] \int_b^c T r dr$$

$$+ \frac{(1 + \nu_c) \alpha_c}{r} \int_b^r T r dr$$

$$w_c = \left[-\frac{2 \nu_c b^2 p'}{E_c (c^2 - b^2)} - \frac{2 \nu_c \alpha_c}{c^2 - b^2} \int_b^c T r dr + (1 + \nu_c) \alpha_c T \right] z$$

$$u_c = \frac{b^2 p'}{h E_c} + \alpha_c b T$$

$$w_c = \left[-\frac{\nu_c b p'}{h E_c} + \alpha_c T \right] z$$

} (FOR THIN CASE)

$$\Delta T - 6$$

TEMPERATURE - SOLID OR HOLLOW CYLINDER - WITH CASE

PLANE STRAIN:

BOUNDARY CONDITIONS:

$$\sigma_r = 0 : r = a, c$$

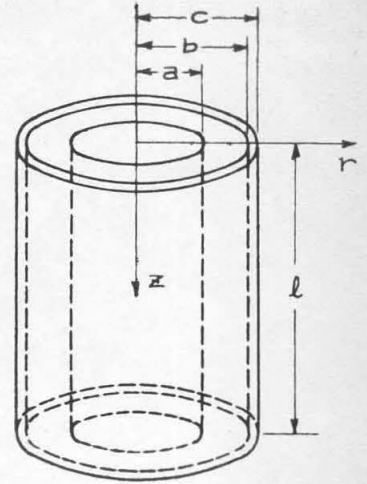
$$w = 0$$

STRESSES IN PROPELLANT:

$$\sigma_r = -\frac{b^2 p'}{(b^2 - a^2)} \left(1 - \frac{a^2}{r^2}\right) + \frac{\alpha E}{(1-\nu)(b^2 - a^2)} \left(1 - \frac{a^2}{r^2}\right) \int_a^b T r dr - \frac{\alpha E}{(1-\nu) r^2} \int_a^r T r dr$$

$$\sigma_\theta = -\frac{b^2 p'}{(b^2 - a^2)} \left(1 + \frac{a^2}{r^2}\right) + \frac{\alpha E}{(1-\nu)(b^2 - a^2)} \left(1 + \frac{a^2}{r^2}\right) \int_a^b T r dr + \frac{\alpha E}{(1-\nu) r^2} \int_a^r T r dr - \frac{\alpha E T}{(1-\nu)}$$

$$\sigma_z = -\frac{2\nu b^2 p'}{(b^2 - a^2)} + \frac{2\nu \alpha E}{(1-\nu)(b^2 - a^2)} \int_a^b T r dr - \frac{\alpha \nu E T}{(1-\nu)}$$



STRAINS & DISPLACEMENTS IN PROPELLANT:

$$\epsilon_r = -\frac{(1+\nu) b^2 p'}{E(b^2 - a^2)} \left[(1-2\nu) - \frac{a^2}{r^2} \right] + \frac{(1+\nu) \alpha}{(1-\nu)(b^2 - a^2)} \left[(1-2\nu) - \frac{a^2}{r^2} \right] \int_a^b T r dr - \frac{\alpha(1+\nu)}{(1-\nu) r^2} \int_a^r T r dr + \frac{(1+\nu^2)}{(1-\nu)} \alpha T$$

$$\epsilon_\theta = -\frac{(1+\nu) b^2 p'}{E(b^2 - a^2)} \left[(1-2\nu) + \frac{a^2}{r^2} \right] + \frac{(1+\nu) \alpha}{(1-\nu)(b^2 - a^2)} \left[(1-2\nu) + \frac{a^2}{r^2} \right] \int_a^b T r dr + \frac{(1+\nu) \alpha}{(1-\nu) r^2} \int_a^r T r dr - \nu \alpha T$$

$$u = -\frac{(1+\nu) b^2 p'}{E(b^2 - a^2)} \left[(1-2\nu) r + \frac{a^2}{r} \right] + \frac{(1+\nu) \alpha}{(1-\nu)(b^2 - a^2)} \left[(1-2\nu) r + \frac{a^2}{r} \right] \int_a^b T r dr + \frac{(1+\nu) \alpha}{(1-\nu) r} \int_a^r T r dr - \nu \alpha T r$$

TEMPERATURE - SOLID OR HOLLOW CYLINDER - WITH CASE

$$p' = \frac{\frac{2\alpha(1+\nu)}{(b^2-a^2)} \int_a^b T r dr - \nu \alpha T - \alpha_c T}{\frac{(1+\nu)[(1-2\nu)b^2+a^2]}{(b^2-a^2)E} + \frac{(1-\nu_c^2)b}{hE_c}} \quad (\text{FOR THIN CASE})$$

PLANE STRAIN COND'D.:STRESSES IN CASE:

$$\sigma_r = \frac{b^2 p'}{(c^2 - b^2)} \left(1 - \frac{c^2}{r^2}\right) + \frac{\alpha_c E_c}{(1 - \nu_c)(c^2 - b^2)} \left(1 - \frac{b^2}{r^2}\right) \int_b^c T r dr - \frac{\alpha_c E_c}{(1 - \nu_c) r^2} \int_b^r T r dr$$

$$\sigma_\theta = \frac{b^2 p'}{(c^2 - b^2)} \left(1 + \frac{c^2}{r^2}\right) + \frac{\alpha_c E_c}{(1 - \nu_c)(c^2 - b^2)} \left(1 + \frac{b^2}{r^2}\right) \int_b^c T r dr + \frac{\alpha_c E_c}{(1 - \nu_c) r^2} \int_b^r T r dr - \frac{\alpha_c E_c T}{(1 - \nu_c)}$$

$$\sigma_z = \frac{2\nu b^2 p'}{(c^2 - b^2)} + \frac{2\nu \alpha_c E_c}{(1 - \nu_c)(c^2 - b^2)} \int_b^c T r dr - \frac{\alpha_c \nu_c E_c T}{(1 - \nu_c)}$$

STRAINS & DISPLACEMENTS IN CASE:

$$\epsilon_r = \frac{(1 + \nu_c) b^2 p'}{E_c (c^2 - b^2)} \left[(1 - 2\nu_c) - \frac{c^2}{r^2} \right] + \frac{(1 + \nu_c) \alpha_c}{(1 - \nu_c)(c^2 - b^2)} \left[(1 - 2\nu_c) - \frac{c^2}{r^2} \right] \int_b^c T r dr - \frac{(1 + \nu_c) \alpha_c}{(1 - \nu_c) r^2} \int_b^r T r dr + \frac{(1 + \nu_c^2)}{(1 - \nu_c)} \alpha_c T$$

$$\epsilon_\theta = \frac{(1 + \nu_c) b^2 p'}{E_c (c^2 - b^2)} \left[(1 - 2\nu_c) + \frac{c^2}{r^2} \right] + \frac{\alpha_c (1 + \nu_c)}{(c^2 - b^2)(1 - \nu_c)} \left[(1 - 2\nu_c) + \frac{c^2}{r^2} \right] \int_b^c T r dr + \frac{(1 + \nu_c) \alpha_c}{(1 - \nu_c) r^2} \int_b^r T r dr - \nu_c \alpha_c T$$

$$u_c = \frac{(1 + \nu_c) b^2 p'}{E_c (c^2 - b^2)} \left[(1 - 2\nu_c) r + \frac{c^2}{r} \right] + \frac{\alpha_c (1 + \nu_c)}{(c^2 - b^2)(1 - \nu_c)} \left[(1 - 2\nu_c) r + \frac{c^2}{r} \right] \int_b^c T r dr + \frac{(1 + \nu_c) \alpha_c}{(1 - \nu_c) r} \int_b^r T r dr - \nu_c \alpha_c T r$$

$$u_c = \frac{(1 - \nu_c^2) b^2 p'}{h E_c} + \alpha_c b T \quad (\text{FOR THIN CASE})$$

TEMPERATURE-UNIFORM-HOLLOW CYLINDER-WITH CASE & ENDS BONDED

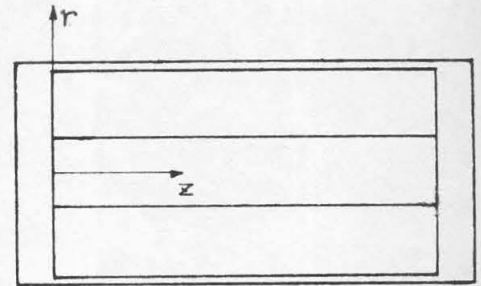
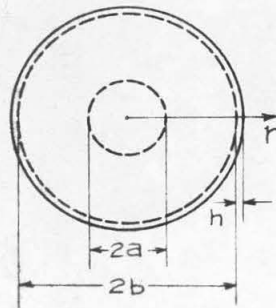
BOUNDARY CONDITIONS:

$$\sigma_r = 0 : r = a$$

$$\sigma_r = 0 : r = b + h$$

$$u = u_c : r = b$$

$$w = w_c : z = 0, l$$



STRESSES IN HOLLOW CYLINDER (PROPELLANT):

$$\sigma_{(r)} = \pm \frac{a^2 b^2 p_0}{b^2 - a^2} \frac{1}{r^2} - \frac{p_0 b^2}{b^2 - a^2},$$

p_0 : PRESSURE BETWEEN CASE & PROPELLANT

$$\sigma_z = -p',$$

p' : PRESSURE BETWEEN ENDS & PROPELLANT

$$p_0 = \frac{[(1+\nu)\epsilon_c^2 + (1+\nu_c)\epsilon_c] \left(\frac{b^2}{a^2} - 1\right) (\alpha - \alpha_c) ET}{(1+\nu) \left\{ (1-2\nu) \frac{b^2}{a^2} + 1 \right\} \epsilon_c^2 + \left\{ [3-\nu(1+4\nu_c)] \frac{b^2}{a^2} + (1+\nu) \right\} \epsilon_c + 2(1-\nu_c^2) \frac{b^2}{a^2}}$$

$$p' = \frac{\left[(1+\nu) \left(\frac{b^2}{a^2} - 1\right) \left(\frac{b^2}{a^2} + 1\right) \epsilon_c^2 + 2(1+\nu_c) \frac{b^2}{a^2} \left(\frac{b^2}{a^2} - 1\right) \epsilon_c \right] (\alpha - \alpha_c) ET}{\left[\frac{b^2}{a^2} - 1 \right] \left\{ (1+\nu) \left\{ (1-2\nu) \frac{b^2}{a^2} + 1 \right\} \epsilon_c^2 + \left\{ [3-\nu(1+4\nu_c)] \frac{b^2}{a^2} + (1+\nu) \right\} \epsilon_c + 2(1-\nu_c^2) \frac{b^2}{a^2} \right\}}$$

WHERE

$$\epsilon_c = \frac{2bh E_c}{(b^2 - a^2) E} \quad (\text{THIN CASE})$$

STRAINS & DISPLACEMENTS IN PROPELLANT:

$$\epsilon_{(r)} = \frac{1}{E} \left[\pm (1+\nu) \frac{a^2 b^2 p_0}{b^2 - a^2} \frac{1}{r^2} - (1-\nu) \frac{p_0 b^2}{b^2 - a^2} + \nu p' \right] + \alpha T$$

$$\epsilon_z = - \frac{1}{E} \left[p' - 2\nu \frac{p_0 b^2}{b^2 - a^2} \right] + \alpha T$$

$$u = \frac{1}{E} \left[-(1+\nu) \frac{a^2 b^2 p_0}{b^2 - a^2} \frac{1}{r} - (1-\nu) \frac{p_0 b^2}{b^2 - a^2} r + \nu p' r \right] + \alpha T r$$

$$w = - \frac{1}{E} \left[p' - 2\nu \frac{p_0 b^2}{b^2 - a^2} \right] z + \alpha T z$$

STRESSES IN CASE:

$$\sigma_{(r)} = \frac{b p_0}{2h} \left(1 \mp \frac{b^2}{r^2} \right) \quad (\text{THIN CASE})$$

$$\sigma_z = p'' = \frac{(b^2 - a^2)p' + a^2 p_i}{2bh}$$

STRAINS & DISPLACEMENTS IN CASE:

$$\epsilon_{(r)} = \frac{1}{E_c} \left[\mp (1+\nu_c) \frac{b^3 p_0}{2h} \frac{1}{r^2} + (1-\nu_c) \frac{b p_0}{2h} - \nu_c p'' \right] + \alpha_c T$$

$$\epsilon_z = \frac{1}{E_c} \left[p'' - \nu_c \frac{b p_0}{h} \right] + \alpha_c T$$

$$u_c = \frac{1}{E_c} \left[(1+\nu_c) \frac{b^3 p_0}{2h} \frac{1}{r} + (1-\nu_c) \frac{b p_0}{2h} r - \nu_c p'' r \right] + \alpha_c T r$$

$$w_c = \frac{1}{E_c} \left[p'' - \nu_c \frac{b p_0}{h} \right] z + \alpha_c T z$$

ASSUMPTIONS:

1. ENDS & CASE BONDED TO PROPELLANT.
2. UNIFORM END EFFECTS WITH END PLATES TRANSMITTING FORCES FROM p_i & p' INTO THIN CYLINDRICAL CASE IN TENSION.
3. NO BENDING EFFECTS INCLUDED FROM ENDS; ASSUMING ONLY HOOP (σ_θ) & AXIAL (σ_z) STRESSES UNIFORM IN CASE UP TO ENDS.
4. UNIFORM TEMPERATURE CHANGE SAME IN PROPELLANT & CASE.

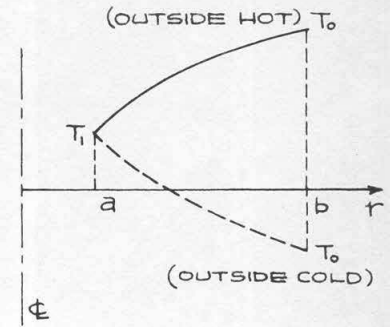
TEMPERATURE - STEADY FLOW - PLANE STRESS - NO CASETEMPERATURE DISTRIBUTION:

$$T = T_0 + \frac{T_1 - T_0}{\log(\frac{b}{a})} \log(\frac{b}{r}), \quad a \leq r \leq b$$

STRESSES:

$$\sigma_r = \frac{\alpha E (T_1 - T_0)}{2 \log \frac{b}{a}} \left\{ -\log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right) \log \frac{b}{a} \right\}$$

$$\sigma_\theta = \frac{\alpha E (T_1 - T_0)}{2 \log \frac{b}{a}} \left\{ 1 - \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right) \log \frac{b}{a} \right\}$$

STRAINS & DISPLACEMENTS:

$$\epsilon_r = \frac{\alpha (T_1 - T_0)}{2 \log \frac{b}{a}} \left\{ -\nu + (1 + \nu) \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left[(1 - \nu) - (1 + \nu) \frac{b^2}{r^2} \right] \log \frac{b}{a} + \frac{2 T_0}{T_1 - T_0} \log \frac{b}{a} \right\}$$

$$\epsilon_\theta = \frac{\alpha (T_1 - T_0)}{2 \log \frac{b}{a}} \left\{ 1 + (1 + \nu) \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left[(1 - \nu) + (1 + \nu) \frac{b^2}{r^2} \right] \log \frac{b}{a} + \frac{2 T_0}{T_1 - T_0} \log \frac{b}{a} \right\}$$

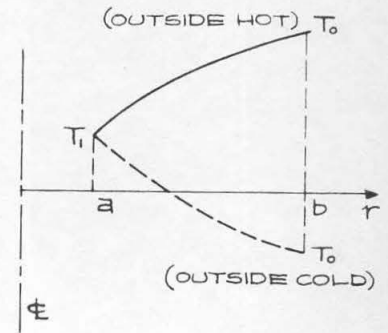
$$\epsilon_z = \frac{\alpha (T_1 - T_0)}{2 \log \frac{b}{a}} \left\{ -\nu + 2(1 + \nu) \log \frac{b}{r} + \frac{2 \nu a^2}{b^2 - a^2} \log \frac{b}{a} + \frac{2 T_0}{T_1 - T_0} \log \frac{b}{a} \right\}$$

$$u = \frac{\alpha (T_1 - T_0)}{2 \log \frac{b}{a}} \left\{ r + (1 + \nu) r \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left[(1 - \nu) r + (1 + \nu) \frac{b^2}{r} \right] \log \frac{b}{a} + \frac{2 T_0 r}{T_1 - T_0} \log \frac{b}{a} \right\}$$

$$w = \frac{\alpha (T_1 - T_0)}{2 \log \frac{b}{a}} \left\{ -\nu + 2(1 + \nu) \log \frac{b}{r} + \frac{2 \nu a^2}{b^2 - a^2} \log \frac{b}{a} + \frac{2 T_0}{T_1 - T_0} \log \frac{b}{a} \right\} z$$

TEMPERATURE-STEADY FLOW-PLANE STRAIN-NO CASETEMPERATURE DISTRIBUTION:

$$T = T_0 + \frac{T_1 - T_0}{\log \frac{b}{a}} \log \frac{b}{r}, \quad a \leq r \leq b$$

STRESSES:

$$\sigma_r = \frac{\alpha E (T_1 - T_0)}{2(1-\nu) \log \frac{b}{a}} \left[-\log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right) \log \frac{b}{a} \right]$$

$$\sigma_\theta = \frac{\alpha E (T_1 - T_0)}{2(1-\nu) \log \frac{b}{a}} \left[1 - \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right) \log \frac{b}{a} \right]$$

$$\sigma_z = \frac{\alpha E (T_1 - T_0)}{2(1-\nu) \log \frac{b}{a}} \left[\nu - 2 \log \frac{b}{r} - \frac{2\nu a^2}{b^2 - a^2} \log \frac{b}{a} - \frac{2(1-\nu)T_0}{T_1 - T_0} \log \frac{b}{a} \right]$$

STRAINS & DISPLACEMENTS:

$$\epsilon_r = \frac{\alpha(1+\nu)(T_1 - T_0)}{2(1-\nu) \log \frac{b}{a}} \left\{ -\nu + \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left[(1-2\nu) + \frac{b^2}{r^2} \right] \log \frac{b}{a} + \frac{2(1-\nu)T_0}{T_1 - T_0} \log \frac{b}{a} \right\}$$

$$\epsilon_\theta = \frac{\alpha(1+\nu)(T_1 - T_0)}{2(1-\nu) \log \frac{b}{a}} \left\{ (1-\nu) + \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left[(1-2\nu) - \frac{b^2}{r^2} \right] \log \frac{b}{a} + \frac{2(1-\nu)T_0}{T_1 - T_0} \log \frac{b}{a} \right\}$$

$$u = \frac{\alpha(1+\nu)(T_1 - T_0)}{2(1-\nu) \log \frac{b}{a}} \left\{ (1-\nu)r + r \log \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left[(1-2\nu)r + \frac{b^2}{r} \right] \log \frac{b}{a} + \frac{2(1-\nu)T_0}{T_1 - T_0} r \log \frac{b}{a} \right\}$$

GRAVITY-SOLID OR HOLLOW CYLINDER-WITH CASE-PURE SHEAREQUILIBLIUM:

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \rho g = 0$$

BOUNDARY CONDITIONS:

$$\tau_{rz} = 0 : r = a$$

$$w = 0 : r = b$$

STRESSES:

$$\tau_{rz} = \frac{\rho g}{2} \left(\frac{a^2}{r} - r \right)$$

STRAIN & DISPLACEMENT:

$$\gamma_{rz} = \frac{\rho g}{2\mu} \left(\frac{a^2}{r} - r \right)$$

$$w = \frac{\rho g}{2\mu} \left(\frac{b^2 - r^2}{2} - a^2 \log \frac{b}{r} \right)$$

ASSUMPTIONS:

1. RIGID CASE.
2. NO END EFFECTS OR EQUIVALENTLY AN INFINITELY LONG CYLINDER.

